Two bounded metric spaces might be homeomorphic to each other with one being complete and another not.

Question: Let (X_1, d_1) and (X_2, d_2) be two bounded metric spaces. For $i = 1, 2$, let π_i be the topology on X_i which is induced from d_i. Assume that (X_1, π_1) is homemorphic to (X_2, π_2), and furthur assume that the metric space (X_1, d_1) is complete, can we claim that the metric space (X_2, d_2) is also complete?

Answer: No.

First of all, completeness is something related to metric structure, while homeomorphism is something related to topology structure. If two metric spaces have the same (or equivalent) metric structures, it is easy to show that they surely are homeomorphic to each other if we regard them as topological spaces with topology derived from metric. The other direction, however, is generally not true. As indicated by Problem 7 of HW No. 5 for the course "Real and Complex Analysis – I", while equipped with the ordinary metric and topology, $(0, 1)$ is homeomorphic to $\mathbb{R}$. However, as metric spaces, $\mathbb{R}$ is complete while $(0, 1)$ is not. Note that the metric space $\mathbb{R}$ here is unbounded while the metric space $(0, 1)$ is bounded.

As for this question asked here, it is furthur assumed that two metric spaces (X_1, d_1) and (X_2, d_2) are bounded. Note that by (X, d) is bounded, we mean $\sup_{x,y\in X} d(x, y) < \infty$. In some textbooks, this $\sup_{x,y\in X} d(x, y)$ is also called the diameter of X.

Under this furthur requirement/assumption, the answer is still a no. To show this, we just need to construct a metric space which is bounded and the corresponding topological space is equivalent to that of the ordinary $\mathbb{R}$. That is because if the topology of regular $\mathbb{R}$ is homeomorphic to the topology of regular $(0, 1)$.

The idea is simple, try to construct this metric subspace as a subspace of $l^\infty(\mathbb{N})$, which is an infinite dimensional metric space. A sketchy outline of such construction is given below:

In $l^\infty(\mathbb{N})$, identify $(0, 0, 0, 0, 0, \cdots)$ as $\overline{0}$, $(1, 0, 0, 0, 0, \cdots)$ as $\overline{1}$, $(1, 1, 0, \cdots)$ as $\overline{2}$, $(1, 1, 1, 0, 0, 0 \cdots)$ as $\overline{3}$, $\cdots$. As for values such as $\overline{3.4}$, it is identified with

$$(1 - 0.4) \cdot \overline{3} + 0.4 \cdot \overline{4} = (1, 1, 1, 0.4, 0, 0, 0, \cdots).$$

Besides, we identify $\overline{-1}$ with $(-1, 0, 0, 0, 0, 0, \cdots)$, $\overline{-2}$ with $(-1, -1, 0, 0, 0, 0, \cdots)$, $\overline{-3}$ with $(-1, -1, -1, 0, 0, 0, \cdots)$, $\cdots$. As for $\overline{-2.34}$, it is just identified with $(-1, -1, -0.34, 0, 0, 0, \cdots)$.

To summarize, we constructed points $\overline{n}$ in $l^\infty(\mathbb{N})$ for all $n \in \mathbb{Z}$. Then we defined the segment $\overline{[n, n+1]}$ just to be the segment in linear space $l^\infty(\mathbb{N})$ that connects $\overline{n}$ and $\overline{n+1}$. Consider the subset $\bigcup_{n\in\mathbb{Z}} \overline{[n, n+1]}$ in $l^\infty(\mathbb{N})$ with the metric structure restricted from the norm of $l^\infty(\mathbb{N})$, and we get a metric space, which is denoted as $\overline{\mathbb{R}}$.

For the metric space $\overline{\mathbb{R}}$ above, check the following:

i) The diameter of $\overline{\mathbb{R}}$ is no more than 2.

ii) This metric space $\overline{\mathbb{R}}$ is complete.

iii) Under the topology derived from the metric, the topological space $\overline{\mathbb{R}}$ is homemorphic to $\mathbb{R}$ with the "normal" topology. *Note: As* $\mathbb{R}$ *with the "normal" topology is homeomorphic to* $(0, 1)$ *with the usual topology, the result in iii) just indicates that the topological space* $\overline{\mathbb{R}}$ *is homemorphi to* $(0, 1)$ *with the usual topology.*